§6.2 留数的应用

一、$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$ 型

作变换 $z = e^{i\theta}$，$e^{i\theta} \cdot i\, d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}$，则：

$\cos\theta = \frac{z+\bar{z}}{2} = \frac{z+\frac{1}{z}}{2}$，$\sin\theta = \frac{z-\frac{1}{z}}{2i}$

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right) \frac{1}{iz}\, dz$$

这里 $R(u,v)$ 是二元有理分式函数

例1　计算 $I = \int_0^{2\pi} \frac{1}{1-2p\cos\theta+p^2}\, d\theta \quad (0 \le |p| < 1)$

Sol. 令 $z = e^{i\theta}$，$dz = e^{i\theta} \cdot i\, d\theta = iz\, d\theta$

$$\cos\theta = \frac{z+\frac{1}{z}}{2},\quad I = \oint_{|z|=1} \frac{1}{1-2p\cdot\frac{z+\frac{1}{z}}{2}+p^2} \cdot \frac{1}{iz}\, dz = \frac{1}{i}\oint_{|z|=1} \frac{dz}{-pz^2+(1+p^2)z-p}$$

$$= i\oint_{|z|=1} \frac{dz}{(pz-1)(z-p)} = i \cdot 2\pi i \cdot \operatorname*{Res}_{z=p} f(z)$$

$$= -2\pi \cdot \frac{1}{p^2-1} = \frac{2\pi}{1-p^2}$$

例2　计算 $I = \int_0^{\pi} \frac{\cos mx}{5-4\cos x}\, dx$

Sol. $I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos mx}{5-4\cos x}\, dx$，记 $I_1 = \frac{1}{2}\int_{-\pi}^{\pi} \frac{\sin mx}{5-4\cos x}\, dx$

则 $$I = I + iI_1 = \frac{1}{2}\int_{-\pi}^{\pi} \frac{e^{imx}}{5-4\cos x}\, dx = \frac{1}{2}\oint_{|z|=1} \frac{z^m}{5-4\cdot\frac{z+\frac{1}{z}}{2}} \cdot \frac{1}{iz}\, dz$$

$$= \frac{i}{2}\oint_{|z|=1} \frac{z^m}{2z^2-5z+2}\, dz = \frac{i}{2}\oint_{|z|=1} \frac{z^m}{(2z-1)(z-2)}\, dz$$

$$= \frac{i}{4}\oint_{|z|=1} \frac{z^m}{(z-\frac{1}{2})(z-2)}\, dz = \frac{i}{4} \cdot 2\pi i \operatorname*{Res}_{z=\frac{1}{2}} f(z) = \frac{\pi}{3} \cdot \frac{1}{2^m}$$

二、$\int_{-\infty}^{+\infty}\frac{P_n(z)}{Q_m(z)}dz$ 型

Lem1. 设 $f(z)$ 在扇形的部分区域 $D:\begin{cases}|z|>r\\ \theta_1\le \arg z\le\theta_2\end{cases}$ 上连续，满足 $\lim\limits_{z\to\infty}zf(z)=\lambda$，记 $\Gamma_\rho: z=\rho e^{i\theta}, \theta_1\le\theta\le\theta_2$，则

$$\lim_{\rho\to+\infty}\int_{\Gamma_\rho} f(z)dz=i(\theta_2-\theta_1)\lambda$$

Pf. $\lim\limits_{z\to\infty}zf(z)=\lambda \Longleftrightarrow \lim\limits_{\rho\to+\infty}\rho e^{i\theta}f(\rho e^{i\theta})=\lambda$ 关于 $\theta\in[\theta_1,\theta_2]$ 一致成立

$$\lim_{\rho\to+\infty}\int_{\Gamma_\rho} f(z)dz \xlongequal{z=\rho e^{i\theta}} \lim_{\rho\to+\infty} i\int_{\theta_1}^{\theta_2} f(\rho e^{i\theta})\rho e^{i\theta}d\theta = i\cdot\lim_{\rho\to+\infty}\int_{\theta_1}^{\theta_2} f(\rho e^{i\theta})\rho e^{i\theta}d\theta$$

$$=i\int_{\theta_1}^{\theta_2}\lambda d\theta = i(\theta_2-\theta_1)\lambda$$

Thm1. 设 $P_n(z)$, $Q_m(z)$ 是两个实系数多项式且互质，其幂 m,n 满足 $m-n\geq 2$，则当 $Q_m(z)\neq 0$ 时，

$$\int_{-\infty}^{+\infty}\frac{P_n(z)}{Q_m(z)}dz = 2\pi i\sum_{\operatorname{Im}a_k>0}\operatorname*{Res}_{z=a_k}\frac{P_n(z)}{Q_m(z)}$$

这里 a_k 是 $Q_m(z)=0$ 的根，也是 $\frac{P_n(z)}{Q_m(z)}$ 的孤立奇点。

Pf. $\int_{-\infty}^{+\infty}\frac{P_n(z)}{Q_m(z)}dz=\lim\limits_{R\to+\infty}\int_{-R}^{R}\frac{P_n(z)}{Q_m(z)}dz$

记实轴上从 $z=-R$ 到 $z=R$ 的有向线段为 $[-R,R]$.

记 $\Gamma_R: z=Re^{i\theta}, 0\le\theta\le\pi$，则

$$\int_{\Gamma_R}\frac{P_n(z)}{Q_m(z)}dz+\int_{[-R,R]}\frac{P_n(z)}{Q_m(z)}dz=\oint\frac{P_n(z)}{Q_m(z)}dz$$

当 R 很大时，$\oint\frac{P_n(z)}{Q_m(z)}dz=2\pi i\sum\limits_{\operatorname{Im}a_k>0}\operatorname*{Res}\limits_{z=a_k}\frac{P_n(z)}{Q_m(z)}$

令 $R\to+\infty$ 得 $\lim\limits_{R\to+\infty}\int_{\Gamma_R}\frac{P_n(z)}{Q_m(z)}dz=0$ (Lem 1.)

于是 $\lim\limits_{R\to+\infty}\int_{-R}^{R}\frac{P_n(z)}{Q_m(z)}dz=\int_{[-R,R]}\frac{P_n(z)}{Q_m(z)}dz=2\pi i\sum\limits_{\operatorname{Im}a_k>0}\operatorname*{Res}\limits_{z=a_k}\frac{P_n(z)}{Q_m(z)}$

例3 计算 $\int_0^{+\infty}\frac{1}{x^4+1}dx$

Sol. $z^4+1=0$ 的根：$a_k = e^{\frac{\pi+2k\pi}{4}i}$, $k=0,1,2,3$.

由Thm1., $\int_0^{+\infty}\frac{1}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{x^4+1}dx = \frac{1}{2}\cdot 2\pi i\left(\operatorname*{Res}_{z=a_0}\frac{1}{z^4+1}+\operatorname*{Res}_{z=a_1}\frac{1}{z^4+1}\right)$

$$=\pi i\left(\frac{1}{4z^3}\Big|_{z=a_0}+\frac{1}{4z^3}\Big|_{z=a_1}\right)$$

$$=\pi i\cdot\frac{(-a_0-a_1)}{4} = -\frac{\pi}{4}i\cdot\left(2\cdot\frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}}{4}\pi$$

例4 计算 $I=\int_0^{+\infty}\frac{1}{(1+x^2)^2}dx$

Sol. $I=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{(1+z^2)^2}dz = \frac{1}{2}\cdot 2\pi i\cdot\operatorname*{Res}_{z=i}\frac{1}{(1+z^2)^2} = \pi i\cdot\frac{1}{1!}\left(\frac{1}{(z+i)^2}\right)'\Big|_{z=i}$

$$=\frac{\pi}{4}.$$

~~例5~~

三. $\int_{-\infty}^{+\infty}\frac{P_n(x)}{Q_q(x)}\begin{matrix}\cos mx\\ \sin mx\end{matrix}\,dx$ 型

Lem 2. 设 $f(z)$ 上扇形域的子域 D：$\begin{cases}|z|>r\\ 0\le\arg z\le\pi\end{cases}$ 上连续且 $\lim_{z\to\infty}f(z)=0$，则当 $m>0$ 时记 $\Gamma_\rho: z=\rho e^{i\theta}, 0\le\theta\le\pi$

$$\lim_{\rho\to+\infty}\int_{\Gamma_\rho}f(z)e^{imz}dz=0.$$

Pf. $\forall\varepsilon>0$，由 $\lim_{z\to\infty}f(z)=0$，$\exists R>0$ s.t. $|z|>R$ 时 $|f(z)|<\varepsilon$.

当 $\rho>R$ 时 $\int_{\Gamma_\rho}f(z)e^{imz}dz = i\rho\int_0^{\pi}f(\rho e^{i\theta})\cdot e^{i\theta}\cdot e^{im\rho e^{i\theta}}d\theta$

$$\left|\int_{\Gamma_\rho}f(z)e^{imz}dz\right|\le\rho\varepsilon\int_0^{\pi}\left|e^{im\rho(\cos\theta+i\sin\theta)}\right|d\theta\le\rho\varepsilon\int_0^{\pi}e^{-m\rho\sin\theta}d\theta$$

$$=2\rho\varepsilon\int_0^{\frac{\pi}{2}}e^{-m\rho\sin\theta}d\theta\le2\rho\varepsilon\int_0^{\frac{\pi}{2}}e^{-m\rho\frac{2}{\pi}\theta}d\theta$$

$$=\frac{\cancel{\rho}\varepsilon\pi}{m}(1-e^{-m\rho})<\frac{\varepsilon\pi}{m}$$

于是 $\lim_{\rho\to+\infty}\int_{\Gamma_\rho}f(z)e^{imz}dz=0$.

Thm 2. 设 $P_n(x)$, $Q_p(x)$ 是两个实系数互质多项式, $Q_p(x)\neq 0$, 则当 $p>n$ 且 $m>0$ 时

$$\int_{-\infty}^{+\infty}\frac{P_n(x)}{Q_p(x)}e^{imx}dx=2\pi i\sum_{\operatorname{Im}a_k>0}\operatorname{Res}_{z=a_k}\frac{P_n(z)}{Q_p(z)}e^{imz}$$

这里 a_k 是 $\frac{P_n(z)}{Q_p(z)}e^{imz}$ 的孤立奇点.

Pf. 记实轴上从 $x=-R$ 到 $x=R$ 的有向线段为 $[-R,R]$.

记 $\Gamma_\rho: z=\rho e^{i\theta}\ (0\le\theta\le\pi)$, 则

$$\int_{\Gamma_\rho}\frac{P_n(z)}{Q_p(z)}e^{imz}dz+\int_{[-R,R]}\frac{P_n(x)}{Q_p(x)}e^{imx}dx=\oint\frac{P_n(z)}{Q_p(z)}e^{imz}dz$$

当 ρ 很大时, $\oint\frac{P_n(z)}{Q_p(z)}e^{imz}dz=2\pi i\sum\limits_{\operatorname{Im}a_k>0}\operatorname{Res}\limits_{z=a_k}\frac{P_n(z)}{Q_p(z)}e^{imz}$

令 $\rho\to+\infty$, $\int_{\Gamma_\rho}\frac{P_n(z)}{Q_p(z)}e^{imz}dz=0$, $\int_{[-R,R]}\frac{P_n(x)}{Q_p(x)}e^{imx}dx\to\int_{-\infty}^{+\infty}\frac{P_n(x)}{Q_p(x)}e^{imx}dx$.

Rem. Thm 2. 中 $e^{imx}=\cos mx+i\sin mx$, 内含两个广义积分:

$$\int_{-\infty}^{+\infty}\frac{P_n(x)}{Q_p(x)}\begin{matrix}\cos mx\\ \sin mx\end{matrix}\,dx$$

例5 计算 $I=\int_{0}^{+\infty}\frac{x\sin x}{x^2+1}dx$

Sol. 记 $I_1=\int_{-\infty}^{+\infty}\frac{x\cos x}{x^2+1}dx=0$ (奇函数)

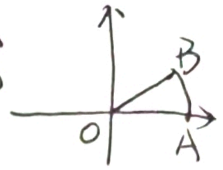
$$I_1+2iI=\int_{-\infty}^{+\infty}\frac{x}{x^2+1}e^{ix}dx=2\pi i\operatorname{Res}_{z=i}\frac{z}{z^2+1}e^{iz}=\frac{\pi i}{e}$$

比较虚部, 有 $I=\frac{\pi}{2e}$

例6 计算 $I=\int_{-\infty}^{+\infty}\frac{\cos mx}{x^2+1}dx=\pi e^{-m}$

四. $\int_0^{+\infty}\cos x^2\,dx$ 与 $\int_0^{+\infty}\sin x^2\,dx$ 计算法

考察 $f(z)=e^{-z^2}$ 在围线 C 上积分，这里 $C=\overrightarrow{OA}+\Gamma_R+\overrightarrow{BO}$



由 Cauchy 积分定理，

$$\oint_C f(z)\,dz=\int_{\overrightarrow{OA}}e^{-z^2}dz+\int_{\Gamma_R}e^{-z^2}dz+\int_{\overrightarrow{BO}}e^{-z^2}dz=0.$$

其中 $\int_{\overrightarrow{OA}}e^{-z^2}dz=\int_0^R e^{-x^2}dx$，$\left|\int_{\Gamma_R}e^{-z^2}dz\right|=\left|\int_0^{\frac{\pi}{4}}Ri\cdot e^{-R^2e^{i2\theta}}\cdot e^{i\theta}d\theta\right|$

$$\le R\cdot\int_0^{\frac{\pi}{4}}e^{-R^2\cos 2\theta}d\theta\xlongequal{2\theta=\frac{\pi}{2}-\varphi}\frac{R}{2}\int_0^{\frac{\pi}{2}}e^{-R^2\sin\varphi}d\varphi$$

$$<\frac{R}{2}\int_0^{\frac{\pi}{2}}e^{-R^2\frac{2}{\pi}\varphi}d\varphi=\frac{\pi}{4R}(1-e^{-R^2})<\frac{\pi}{4R}$$

$$\int_{\overrightarrow{BO}}e^{-z^2}dz\xlongequal{z=xe^{i\frac{\pi}{4}}}\int_R^0 e^{\frac{\pi}{4}i}\cdot e^{-x^2i}dx=-e^{\frac{\pi}{4}i}\left[\int_0^R\cos x^2dx-i\int_0^R\sin x^2dx\right]$$

令 $R\to+\infty$ 得，$\int_0^{+\infty}e^{-x^2}dx+0-e^{\frac{\pi}{4}i}\left[\int_0^{+\infty}\cos x^2dx-i\int_0^{+\infty}\sin x^2dx\right]=0$

即 $\int_0^{+\infty}\cos x^2dx-i\int_0^{+\infty}\sin x^2dx=e^{-\frac{\pi}{4}i}\cdot\frac{\sqrt{\pi}}{2}=\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\cdot\frac{\sqrt{\pi}}{2}$

比较实虚部，有 $\int_0^{+\infty}\cos x^2dx=\int_0^{+\infty}\sin x^2dx=\frac{\sqrt{2\pi}}{4}$

Rem. 还可用留数定理计算 $\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$